# Triangular Numbers

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## Teacher Notes & Answers

7 8 9 10 11 **12**

TI-30XPlus MathPrint™ | Activity | Student | 30 min

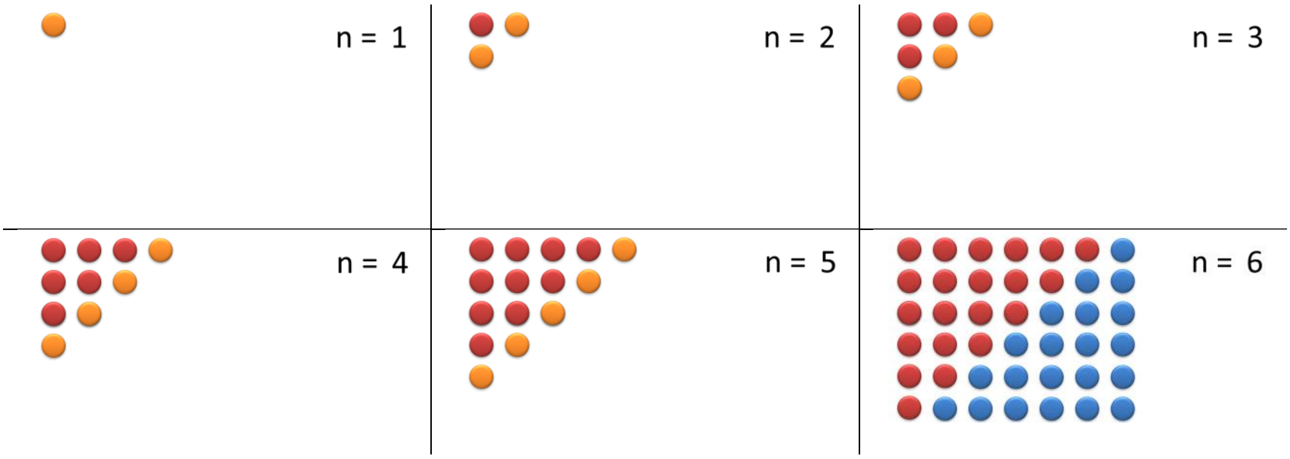
## Introduction to Induction

What is the sum of the first *n* whole numbers?

There are several ways this problem can be solved. Given any value for *n* you could add the numbers up, one by one, but if *n* is large this could take a lot of time. A formula would be a much quicker way to determine such a sum. In this activity you will work with a range of visual and numerical methods to arrive at a formula. However, the formula is based on observation, intuition and a relatively small sample of numbers. There are many cases where formulas seemed to work, but are later found to be flawed. In the final stage of this activity you will prove that your formula works for all whole number values for *n*.

## Visual Observation

The series of diagrams below shows one way to visual sums of the first *n* whole numbers. In each case the new row (orange) shows the quantity being added. The diagrams show why the pattern is referred to as 'triangular' numbers. The last representation includes a duplication of the pattern.




### Question: 1.

The following questions refer to the last pattern (n = 6).

a) How many dots in the last pattern?

b) Explain how you determined this quantity.

c) What is the sum of the first 6 whole numbers?

### Question: 2.

Determine the sum of the first 7 whole numbers without using 'addition'.

### Question: 3.

Generalise your answer to Question 2 for the sum of the first *n* whole numbers.



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### Question: 4.

Use your formula (above) to calculate the sum of the first 100 whole numbers.

### Question: 5.

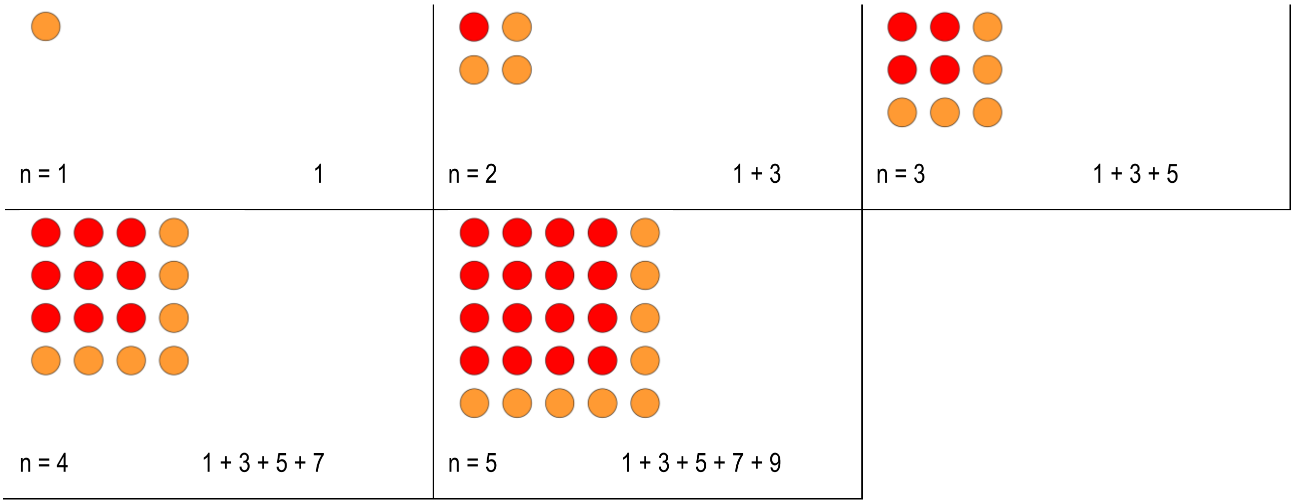
Use the sum command on your calculator to determine the sum of the first 100 numbers.

Expression: $\sum_{x=1}^{100} x$ Calculator instructions: Press **[math]** and select option: **5** sum(.

## Visual Observation + Numerical Intuition:

In this section you will study the sum of the first *n* odd numbers, then the first *n* even numbers and finally derive a formula for the sum of the first *n* whole numbers.

The following sequence of images represents the sum of the first *n* odd numbers.




### Question: 6.

The following questions relate to the sum of the first *n* odd numbers.

a) What shape can be formed by the sum of the first *n* odd numbers?

b) Write a formula for the sum of the first *n* odd umbers.

c) Use your calculator to check the sum of the first 50 odd numbers by using: $\sum_{x=1}^{50} 2x-1$

### Question: 7.

The sum of the first n even numbers can be determined by comparing with the sum of the first n odd numbers.

1 + 3 + 5 + 7 + 9 + 11 + 13 …

2 + 4 + 6 + 8 + 10 + 12 + 14 …

Notice that each of the even numbers is '1' more than the corresponding odd number.

a) Based on this information, determine the sum of the first 50 even numbers.

b) Write a formula for the sum of the first *n* even numbers.

c) Use your calculator to check the sum of the first 50 even numbers by using: $\sum_{x=1}^{50} 2x$

d) If sum of the first *n* even numbers: 2 + 4 + 6 + …is divided 2, the result is 1 + 2 + 3 … hence write a formula for the sum of the first n whole numbers.



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## Pascal's Triangle – Hidden Gem

Pascal's triangle also contains the triangular numbers.

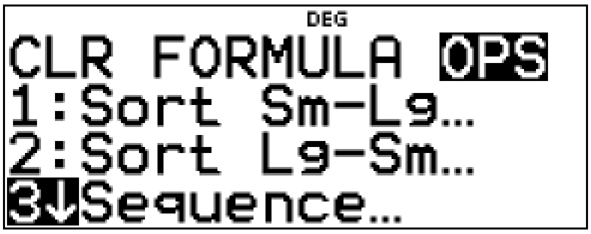
**Notice:** The $n^{th}$ triangular number is in the $(n+1)^{th}$ row[1].

**Example**: The number 15 is the $5^{th}$ triangular number and it is located in the $6^{th}$ row.

Recall that the elements in Pascal's triangle can be computed using combinatorics: ${}^{n}C_{r} = \dfrac{n!}{(n-r)!r!}$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

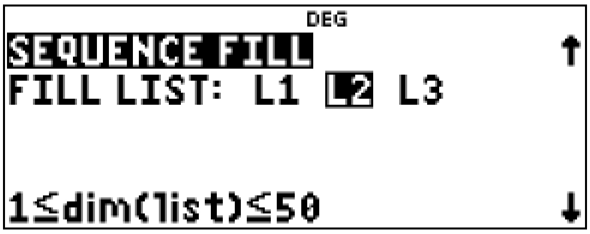
Based on this information, the calculator can be used to generate the triangular numbers.


Select option 3 – Sequence:


The list of triangular numbers will be stored in List 2.

Select List 2:


The expression for List 2 is: ${}^{x}C_{2}$

**Expression**:



**Start:**


**End**:


**Step Size:**


Navigate down to Sequence Fill:


This will calculate: ${}^{2}C_{2}$, ${}^{3}C_{2}$, ${}^{4}C_{2}$, ${}^{5}C_{2}$ … ${}^{10}C_{2}$

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[1] Row numbering in Pascal's triangle starts at row(0) = {1}, row(1) = {1, 1}, row(2) = {1, 2, 1}

## Question: 8.

a) Use your calculator to write down the first 9 triangular numbers. [Store the values in $L_2$]

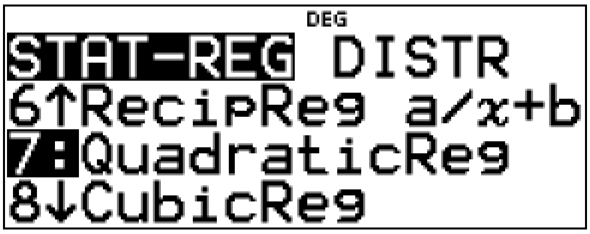
b) Use combinatorics to calculate the 100th triangular number, the sum of the first 100 whole numbers.

c) Store the numbers: {1, 2, 3... 9} in $L_1$ and use Quadratic regression to determine an equation relating $L_1$ to $L_2$. Write down the regression equation. [Calculator Instructions Below]

Check to make sure $L_1$ and $L_2$ are the same length and corresponding values are aligned.

Access the statistics menu:


Scroll down to Quadratic Regression: (Option 7)

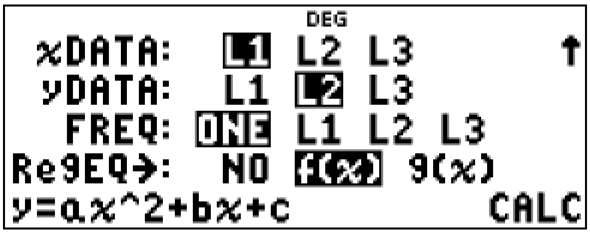



Match the selections shown opposite then select "CALC".

Note the format for the equation:

$$y = ax^2 + bx + c$$

The regression equation will be stored in $f(x)$.




The equation stored in $f(x)$ can be tested by generating a table of values.

Press the **[Table]** key and select option 1: Add/Edit func. The function will be displayed.

Use the arrow keys to navigate down through the menu to 'calc' then press **[Enter]** to generate the table.

d) The diagonal for the triangular numbers can be written using combinatorics: ${}^{n+1}C_2 = \dfrac{(n+1)!}{((n+1)-2!)2!}$

Simplify this formula to write an expression for the $n^{th}$ triangular number.

# Induction

The formula for the sum of the first n whole numbers has been generated three different ways. In each case the formula has been based on 'observation', not proof.

**Step 1:** Show true for $n = 1$.

We must first prove that the formula is true for $n = 1$.

The sum of the first '1' whole numbers is equal to 1 and $\dfrac{n(n+1)}{2} = \dfrac{1 \times 2}{2} = 1$

**Step 2**: Assume true for $n$

That is: $1 + 2 + 3 + \ldots n = \dfrac{n(n+1)}{2}$ **-- Equation 1**



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**Step 3**: Show true for $n$ + 1.

We know that the LHS = (1 + 2 + 3 + … $n$) + ($n$ + 1),

From Equation 1 we can re-write this as: $\frac{n(n+1)}{2}+(n+1)$

## Question: 9.

Complete step 3 by re-writing the RHS: $\frac{(n+1)(n+1+1)}{2}$ to show that: $\frac{n(n+1)}{2}+n+1=\frac{(n+1)(n+1+1)}{2}$



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